Electronic Journal of Qualitative Theory of Differential Equations
2012, No. **3**, 1-11; **http://www.math.u-szeged.hu/ejqtde/**

ON SINGULAR SOLUTIONS FOR SECOND ORDER DELAYED DIFFERENTIAL EQUATIONS

MIROSLAV BARTUŠEK

Abstract. Asymptotic properties and estimate of singular solutions (either defined on a finite interval only or trivial in a neighbourhood of ∞) of the second order delay differential equation with p-Laplacian are investigated.

1. Introduction

In this paper, we consider the second order nonlinear delay differential equation

$$\big(a(t)|y'|^{p-1}y'\big)' + r(t)\big|y(\varphi(t))\big|^{\lambda}\operatorname{sgn} y\big(\varphi(t)\big) = 0 \tag{1}$$

where $p > 0$, $\lambda > 0$, $a \in C^0(\mathbb{R}_+)$, $r \in C^0(\mathbb{R}_+)$, $\varphi \in C^0(\mathbb{R}_+)$, $a(t) > 0$, $r(t) > 0$, $\varphi(t) \le t$ on $\mathbb{R}_+$ and $\lim\limits_{t\to\infty} \varphi(t) = \infty$.

If $p = \lambda$, it is known as the half-linear equation, while if $\lambda > p$, we say that equation (1) is of the super-half-linear type, and if $\lambda < p$, we will say that it is of the sub-half-linear type.

We begin by defining what is mean by a solution of equation (1) as well as some basic properties of solutions.

Definition 1. Let $T \in (0,\infty]$, $\varphi_0 = \inf\limits_{t\in\mathbb{R}_+} \varphi(t)$, $\phi \in C^0[\varphi_0, 0]$, and $y'_0 \in \mathbb{R}$. We say that a function y is a solution of (1) on $[0,T)$ (with the initial conditions (ϕ, y'_0)) if $y \in C^0[\varphi_0, T)$, $y \in C^1[0,T)$, $a|y'|^{p-1}y' \in C^1[0,T)$, (1) holds on $[0,T)$, $y(t) = \phi(t)$ on $[\varphi_0, 0]$, and $y'_+(0) = y'_0$.

We assume that solutions are defined on their maximal interval of existence to the right.

Equation (1) can be written as the equivalent system

$$\begin{aligned} y_1' &= a^{-\frac{1}{p}}(t)|y_2|^{\frac{1}{p}}\operatorname{sgn} y_2\,,\\ y_2' &= -r(t)\big|y(\varphi(t))\big|^{\lambda}\operatorname{sgn} y\big(\varphi(t)\big)\,. \end{aligned} \tag{2}$$

The relationship between a solution y of (1) and a solution (y_1, y_2) of the system (2) is

$$y_1(t) = y(t) \quad\text{and}\quad y_2(t) = a(t)\big|y'(t)\big|^{p-1}y'(t)\,, \tag{3}$$

and when discussing a solution y of (1), we will often use (3) without mention.

1991 *Mathematics Subject Classification.* 34C10, 34C15, 34D05.

Key words and phrases. singular solutions, noncontinuable solutions, second order equations, p-Laplacian, delay.

Definition 2. Let y be a solution of (1) defined on $[0, T)$, $T \le \infty$. It is called singular of the 1st kind if $T = \infty$, $\tau \in (0, \infty)$ exists such that $y \equiv 0$ on $[\tau, \infty)$ and y is nontrivial in any left neighbourhood of τ. Solution y is called singular of the 2nd kind if $T < \infty$ and put $\tau = T$. It is called proper if $T = \infty$ and it is nontrivial in any neighbourhood of ∞. Singular solutions of either 1st or 2nd kind are called singular.

Note, that a solution of (1) is either proper, or singular or trivial on (φ_0, ∞). Singular solutions of the second kind are sometimes called noncontinuable. When discussing singular solutions, τ will be the number in Definition 2 in all the paper without mention.

Remark 1. If y is a singular solution of (1) of the 2nd kind, then it is defined on $[0, \tau)$, $\tau < \infty$ and it cannot be defined at $t = \tau$; so, $\limsup_{t \to \tau}(|y_1(t)| + |y_2(t)|) = \infty$. From this and from (2)

$$\limsup_{t \to \tau} \big|y_2(t)\big| = \infty \,. \tag{4}$$

Definition 3. Let y be a singular solution of (1) of the 1st kind (of the 2nd kind). Then it is called oscillatory if there exists a sequence of its zeros tending to τ and it is called nonoscillatory otherwise.

Singular solutions of (1) without delay, i.e. of

$$\big(a(t)|y'|^{p-1}y'\big)' + r(t)|y|^{\lambda} \operatorname{sgn} y = 0 \,, \tag{5}$$

have been studied by many authors, see e.g. [1, 5], [9]–[16] and the references therein. Note, that the first existence results are obtained in [12] for $p = 1$, $a = 1$ and $r \le 0$. In the monography of Kiguradze and Chanturia [13] it is a good overview of results for $p = 1$ and $a = 1$.

Eq. (5) may have singular solutions. Heidel [11] (Coffman, Ulrych [9]) proved the existence of an equation of type (5), $a \equiv 1$, $p = 1$ with singular solutions of the 1st kind (of the 2nd kind) in case $\lambda < p$ ($\lambda > p$); in this case r is continuous but not of locally bounded variation. If a and r are smooth enough, then singular solutions of (5) do not exist (see Theorem A below). As concerns to Eq. (1), the existence of singular solutions of the second kind are investigated in [4] in case $r \le 0$. The existence and properties of singular solutions of either the first kind or of the second kind in case $r \ge 0$ seem not to be studied at all.

The following theorem sums up results concerning to Eq. (5).

Theorem A. *Let $r \in C^0(\mathbb{R}_+)$ and $r(t) > 0$ on $\mathbb{R}_+$.*

(i) *If $\lambda \ge p$, then there exists no singular solution of (5) of the 1st kind.*
(ii) *If $\lambda \le p$, then there exists no singular solution of (5) of the 2nd kind.*
(iii) *If $a^{\frac{1}{p}} r \in C^1(\mathbb{R}_+)$, then all solutions of (5) are proper.*

Proof. (i), (ii): See Theorems 1.1 and 1.2 in [15]. (iii): It follows from Theorem 2 in [5]. □

Note that estimates of such kind of solutions are proved by Kvinikadze, see references in [13]. In [1] (for $p = 1$, $a = 1$, $r \le 0$) precise asymptotic formulas of all solutions are obtained for differential equations of the third and fourth orders, see also [3]. About uniform estimates of solutions of quasi-linear ordinary differential equations see [2]. In [16] estimates of singular solutions of the second kind of a system of second order differential equations (of the form (5)) are derived.

Theorem B ([16], Theorem 2). *Let $r \in C^0(\mathbb{R}_+)$ and $r(t) > 0$ on $\mathbb{R}_+$. Let $\lambda > p$, y be a singular solution of (5) of the second kind, $T \in [0, \tau)$, $\tau - T \le 1$, $r_0 = \max_{T \le s \le \tau} r(s)$, $C_0 = 2^{\lambda+2}$ in case $p > 1$ and $C_0 = 2^{2\lambda+1}$ in case $p \le 1$. Then a positive constant $C = C(p, \lambda, \tau, r_0)$ exists such that*

$$\big|y_2(t)\big| + C_0 r_0 \big|y(t)\big|^{\lambda} \ge C(\tau - t)^{-\frac{p(\lambda+1)}{\lambda - p}}, \qquad t \in [T, \tau).$$

It is important to study the existence of proper/singular solutions. When studying solutions of (1) and (5), some authors sometimes investigate properties of solutions that are defined on $\mathbb{R}_+$ only without proving the existence of them. Moreover, sometimes, proper solutions have crucial role in a definition of some problems, see e.g. the limit-point/limit-circle problem in [6], [8]. Furthermore, noncontinuable solutions appear e.g. in water flow problems (flood waves, a flow in sewerage systems), see e.g. [4].

Our goal is to study properties of singular solutions and to extend Theorems A and B to (1).

For convenience, we define the constants and the function

$$\delta = \frac{p+1}{p}, \quad \gamma = \frac{p+1}{p(\lambda+1)}, \quad R(t) = a^{\frac{1}{p}}(t)\, r(t), \quad t \in \mathbb{R}_+ .$$

If y is a solution of (1), then we set on its interval of existence

$$F(t) = R^{-1}(t)\big|y_2(t)\big|^{\delta} + \gamma\big|y(t)\big|^{\lambda+1}. \tag{6}$$

Notice that $F(t) \ge 0$ for every solution of (1) and

$$F'(t) = -\frac{R'(t)}{R^2(t)}\big|y_2(t)\big|^{\delta} + \delta y'(t)\, e(t) \tag{7}$$

with

$$e(t) \overset{\text{def}}{=} \big|y(t)\big|^{\lambda} \operatorname{sgn} y(t) - \big|y(\varphi(t))\big|^{\lambda} \operatorname{sgn} y\big(\varphi(t)\big). \tag{8}$$

From (6)

$$\begin{aligned} &\big|y(t)\big| \le \big(\gamma^{-1}F(t)\big)^{\frac{1}{\lambda+1}}, \qquad \big|y_2(t)\big| \le \big[R(t)\, F(t)\big]^{\frac{p}{p+1}}, \\ &\big|y'(t)\big| \le a^{-\frac{1}{p}}(t) R^{\frac{1}{p+1}}(t)\, F^{\frac{1}{p+1}}(t). \end{aligned} \tag{9}$$

2. Singular solutions of the 2nd kind

The following theorem shows that such solutions do not exist in case $\lambda \le p$.

Theorem 1. *If $\lambda \le p$, then all solutions of (1) are defined on $\mathbb{R}_+$*

Proof. It is proved in Lemma 7 in [6] for $r < 0$, for arbitrary r the proof is the same, it is necessary to replace r by $|r|$. ☐

The following theorem gives us basic properties.

Theorem 2. *Let y be a singular solution of* (1) *of the second kind. Then it is oscillatory and $\varphi(\tau) = \tau$. If, moreover, $R \in C^1(\mathbb{R}_+)$, then $\varphi(t) \not\equiv t$ in any left neighbourhood of τ.*

Proof. Suppose, contrarily, that $\varphi(\tau) < \tau$. Then an interval $I = [\tau_1, \tau)$ exists such that $\tau_1 < \tau$ and $\sup\limits_{t\in I} \varphi(t) < \tau$. From this and from (1) we have $|y_2'(t)| = r(t)|y(\varphi(t))|^\lambda \le \sup\limits_{t\in I} r(t)|y(\varphi(t))|^\lambda < \infty$. Hence, y_2 is bounded on I that contradicts (4). Hence, $\varphi(\tau) = \tau$.

Let y be nonoscillatory. Suppose, for the simplicity, that y is positive in a left neighbourhood of τ. Then, with respect to $\varphi(\tau) = \tau$, $\tau_1 < \tau$ exists such that

$$y\big(\varphi(t)\big) > 0 \quad \text{on} \quad I \stackrel{\text{def}}{=} [\tau_1, \tau)\,. \tag{10}$$

As according to (2) and (10), y_2 is decreasing on I and (4) implies

$$\lim_{t\to\tau-} y_2(t) = -\infty\,. \tag{11}$$

From this $\tau_2 \in I$ exists such that

$$y'(t) < 0 \quad \text{on} \quad [\tau_2, \tau) \tag{12}$$

and the integration of (1) and (11)

$$\int_{\tau_2}^{\tau} r(t) y^\lambda\big(\varphi(t)\big)\, dt = y_2(\tau_2) - \lim_{t\to\tau-} y_2(t) = \infty\,.$$

Hence, $\limsup\limits_{t\to\tau-} y(t) = \infty$ that contradicts (12) and y is oscillatory.

Let y be a singular solution of (1) and $\varphi(t) \equiv t$ on a left neighbourhood J on τ. Then y is a singular solution of (5) on J. A contradiction with Theorem A(iii) proves that $\varphi(t) \not\equiv t$ in any left neighbourhood of τ. $\square$

Remark 2. According to Theorem 1 there exists no singular solution of (1) of the second kind in case $\varphi(t) < t$ on $\mathbb{R}_+$; all solutions are defined on $\mathbb{R}_+$. This fact was used by many authors for special types of (1), see e.g. [10], [4] ($r < 0$).

The following two lemmas serve us for estimate of solutions.

Lemma 1. *Let $\omega > 1$, $t_0 \in \mathbb{R}_+$, $K > 0$, Q be a continuous nonnegative function on $[t_0, \infty)$ and u be continuous and nonnegative on $[t_0, \infty)$ satisfying*

$$u(t) \le K + \int_{t_0}^{t} Q(s)\, u^\omega(s)\, ds \quad \textit{on} \quad [t_0, T), T \le \infty\,. \tag{13}$$

If

$$(\omega - 1)K^{\omega-1} \int_{t_0}^{\infty} Q(s)\, ds < 1 \tag{14}$$

then

$$u(t) \le K\big[1 - (\omega - 1)K^{\omega-1} \int_{t_0}^{t} Q(s)\, ds\big]^{1/(1-\omega)}, \quad t \in [t_0, T)\,. \tag{15}$$

Proof. It is proved in Lemma 2.1 in [14] for $m = \omega$ and $p = 1$. $\square$

Lemma 2. *Let $\lambda > p$, $\int_0^\infty r(s)\bigl(\int_0^s a^{-\frac{1}{p}}(\sigma)\,d\sigma\bigr)^\lambda ds < \infty$, y be a solution of (1) defined on $[0,T)$, $T \le \infty$ and let $t_0 \in [0,T)$. If $y_* = \max\limits_{\varphi(t_0) \le s \le t_0} |y(s)|$ and*

$$(16)\quad \Bigl[|y_2(t_0)| + 2^\lambda\, y_*^\lambda \int_{t_0}^\infty r(s)\,ds\Bigr]^{\frac{\lambda}{p}-1} \int_{t_0}^\infty r(s)\Bigl(\int_{t_0}^s a^{-\frac{1}{p}}(\sigma)\,d\sigma\Bigr)^\lambda ds < 2^{-\lambda}\frac{p}{\lambda - p}\,.$$

Then $T = \infty$ and y is defined on $\mathbb{R}_+$.

Proof. Suppose, contrarily, that y is singular of the 2nd kind. Then $T = \tau < \infty$ and denote by

$$v(t) = \sup_{t_0 \le s \le t} |y_2(s)| \quad \text{for} \quad t \in I \overset{\text{def}}{=} [t_0, T)\,.$$

It follows from (2) that

$$|y_2(t)| \le |y_2(t_0)| + \int_{t_0}^t r(s)\,|y\bigl(\varphi(s)\bigr)|^\lambda\,ds$$

and

$$|y(t)| \le |y(t_0)| + \int_{t_0}^t a^{-\frac{1}{p}}(s)|y_2(s)|^{\frac{1}{p}}\,ds\,, \qquad t \in I\,.$$

Hence, for $t_0 \le s \le t < T$ we have

$$\begin{aligned} |y_2(s)| &\le |y_2(t_0)| + \int_{t_0}^s r(z)\Bigl[y_* + v^{\frac{1}{p}}(z)\int_{t_0}^z a^{-\frac{1}{p}}(\sigma)\,d\sigma\Bigr]^\lambda dz \\ &\le |y_2(t)| + 2^\lambda y_*^\lambda \int_{t_0}^\infty r(\sigma)\,d\sigma + 2^\lambda \int_{t_0}^t r(z)\Bigl(\int_{t_0}^z a^{-\frac{1}{p}}(\sigma)\,d\sigma\Bigr)^\lambda v^{\frac{\lambda}{p}}(z)\,dz\,. \end{aligned}$$

From this

$$(17)\quad v(t) \le |y_2(t_0)| + 2^\lambda y_*^\lambda \int_{t_0}^\infty r(\sigma)\,d\sigma + 2^\lambda \int_{t_0}^t r(z)\Bigl(\int_{t_0}^z a^{-\frac{1}{p}}(\sigma)\,d\sigma\Bigr)^\lambda v^{\frac{\lambda}{p}}(z)\,dz\,.$$

Put $\quad \omega = \frac{\lambda}{p} > 1,\ \ u = v,\ \ K = |y_2(t_0)| + 2^\lambda y_*^\lambda \int\limits_{t_0}^\infty r(s)\,ds$

and $\quad Q(t) = 2^\lambda r(t)\Bigl(\int\limits_{t_0}^t a^{-\frac{1}{p}}(\sigma)\,d\sigma\Bigr)^\lambda$.

Then (16) and (17) imply (13) and (14), and according to Lemma 1, (15) is valid. As $T < \infty$, y_2 is bounded on J. A contradiction with (4) proves the statement. $\square$

Remark 3. Note that Lemma 2 is valid even if we suppose $r \ge 0$ instead of $r > 0$ on $\mathbb{R}_+$.

Remark 4. The idea of the proof is due to Medveď and Pekárková [14] (with $\varphi(t) \equiv t$); it is used also in [7] for (1) with $t - \varphi(t) \le$ const. on $\mathbb{R}_+$.

The next theorem derives an estimate from below of a singular solution of the second kind.

Theorem 3. *Let $\lambda > p$ and let y be a singular solution of* (1) *of the 2nd kind. Let $T \in [0, \tau)$, $a_* = \min_{T \le s \le \tau} a(s)$, $r_* = \max_{T \le s \le \tau} r(s)$ and $y_*(t) = \max_{\varphi(t) \le s \le t} |y(s)|$ on $[T, \tau)$. Then*

$$|y_2(t)| + 2^{\lambda+1} y_*^\lambda(t)\, r_*(\tau - t) \ge K(\tau - t)^{\frac{-p(\lambda+1)}{\lambda-p}} \tag{18}$$

on $[T, \tau)$ with $K = \left(2^{-2\lambda-1} \frac{(\lambda+1)p}{\lambda-p} a_*^{\frac{\lambda}{p}} r_*^{-1}\right)^{\frac{p}{\lambda-p}}$. Especially, a left neighbourhood I of τ exists such that

$$a(\tau)|y'(t)|^p + 2^{\lambda+1} y_*^\lambda(t)\, r(\tau)(\tau - t) \ge K_1(\tau - t)^{\frac{-p(\lambda+1)}{\lambda-p}} \tag{19}$$

on I with $K_1 = \left[2^{-2\lambda-3-\frac{\lambda}{p}} \frac{(\lambda+1)p}{\lambda-p} a^{\frac{\lambda}{p}}(\tau) r^{-1}(\tau)\right]^{\frac{p}{\lambda-p}}$.

Proof. Let y be a singular solution of (1) of the 2nd kind defined on $[0, \tau)$. Let $\bar{t} \in [T, \tau)$ be fixed. Define

$$\begin{aligned}
&\bar{r}(t) = r(t) && \bar{a}(t) = a(t) \quad \text{for} \quad t \in [0, \tau]\,,\\
&\bar{r}(t) = \frac{r(\tau)}{\tau - \bar{t}}(-t + 2\tau - \bar{t})\,, && \bar{a}(t) = \frac{a(\tau)}{\tau - \bar{t}}(-t + 2\tau - \bar{t}) \quad \text{for} \quad t \in (\tau, 2\tau - \bar{t}]\\
&\bar{r}(t) = 0\,, && \bar{a}(t) = 0 \quad \text{for} \quad t > 2\tau - \bar{t}\,;
\end{aligned}$$

note that $\bar{r}$ and $\bar{a}$ are continuous on $\mathbb{R}_+$ and are linear on $[\tau, 2\tau - \bar{t}]$. Furthermore, we have

$$\begin{aligned}
\int_{\bar{t}}^{\infty} \bar{r}(s)\Big(\int_{\bar{t}}^{s} \bar{a}^{-\frac{1}{p}}(\sigma)\, d\sigma\Big)^{\lambda} ds &\le r_* a_*^{-\frac{\lambda}{p}} \int_{\bar{t}}^{2\tau-\bar{t}} (s - \bar{t})^{\lambda}\, ds\\
&\le \frac{2^{\lambda+1}}{\lambda+1} r_*\, a_*^{-\frac{\lambda}{p}} (\tau - \bar{t})^{\lambda+1}
\end{aligned} \tag{20}$$

and

$$\int_{\bar{t}}^{\infty} r(s)\, ds \le \int_{\bar{t}}^{2\tau-\bar{t}} r_*\, ds = 2r_*(\tau - \bar{t})\,. \tag{21}$$

Consider an auxilliary equation

$$\big(\bar{a}(t)|z'|^{p-1}z'\big) + \bar{r}(t)|z(\varphi)|^{\lambda} \operatorname{sgn} z(\varphi) = 0\,. \tag{22}$$

Then $z = y$ is the singular solution of (22) of the second kind defined on $[0, \tau)$. Suppose that (18) is not valid for $t = \bar{t}$, i.e.

$$\big[|y_2(\bar{t}) + 2^{\lambda+1} y_*^\lambda(\bar{t})\, r_*(\tau - \bar{t})\big]^{\frac{\lambda}{p}-1} < 2^{-2\lambda-1} \frac{(\lambda+1)p}{\lambda - p} a_*^{\frac{\lambda}{p}}\, r_*^{-1}(\tau - \bar{t})^{-\lambda-1} \tag{23}$$

holds. We apply Lemma 2 and Remark 3 with $T = \tau$ and $t_0 = \bar{t}$. Then it follows from (20), (21) and (23) that all assumptions of Lemma 2 are valid. Hence, z is defined on $\mathbb{R}_+$ and the contradiction with z to be singular proves that (18) is valid. Furthermore, a left neighbourhood I of $t = \tau$ exists such that

$$r_* \le 2r(\tau) \quad \text{and} \quad \frac{a(\tau)}{2} \le a_* \le 2a(\tau)$$

and (20) follows from this and from (18). $\square$

Remark 5. The used method of the proof of Theorem 2 is due to Pekárková [16] (for $\varphi(t) \equiv t$).

Corollary 1. *Every singular solution of* (1) *of the second kind is unbounded.*

Remark 6. In case $\varphi(t) \equiv t$, Theorem 3 gives us similar estimate than Theorem B but it can be used also for $\tau - t > 1$.

Corollary 2. *Let y be a singular solution of* (1) *of the second kind. Then a sequence $\{t_k\}_{k=1}^{\infty}$ of local extremes and constant $M > 0$ exist such that $\lim_{k\to\infty} t_k = \tau$ and*

$$\left|y(t_k)\right| \geq M(\tau - t_k)^{\frac{-p(\lambda+1)}{\lambda(\lambda-p)}}, \qquad k = 1, 2, \dots$$

Proof. Let y be a singular solution of the 2nd kind. Then according to Lemma 2 and Corollary 2 it is oscillatory and unbounded. Hence, an increasing sequence $\{t_k\}_{k=1}^{\infty}$ exists such that $\lim_{t\to\infty} t_k = \tau$, y has the local extreme at t_k and

$$\left|y(t_k)\right| \geq |y(t)| \quad \text{for} \quad t \in [\varphi_0, t_k]\,,\ k = 1, 2, \dots$$

Then $y'(t_k) = 0$, $\max_{\varphi(t_k)\leq s\leq t_k} |y(s)| = |y(t_k)|$, and the statement follows from (19). □

3. Singular solution of the 1st kind

This paragraph begins with some basic properties

Theorem 4. *Let y be a singular solution of* (1) *of the first kind. Then it is oscillatory and $\varphi(\tau) = \tau$. Moreover,*

(i) *if $R \in C^1(\mathbb{R}_+)$, then $\varphi(t) \not\equiv t$ in any left neighbourhood of τ;*

(ii) *if $R \in C^1(\mathbb{R}_+)$, $\lambda \geq p$ and φ is nondecreasing in a left neighbourhood J of τ, then a left neighbourhood J_1 of τ exists such that $\varphi(t) < t$ on J_1.*

Proof. Let y be a singular solution of (1) of the first kind. Then

$$y(t) = 0 \quad \text{for} \quad t \geq \tau \tag{24}$$

and

$$y(t) \not\equiv 0 \quad \text{in any left neighbourhood of } \tau. \tag{25}$$

Suppose, contrarily, that $\varphi(\tau) < \tau$. Then $\lim_{t\to\infty} \varphi(t) = \infty$ implies the existence of τ_1 such that $\tau_1 > \tau$ and $\varphi(t) > \tau$ for $t \geq \tau_1$. Denote $I = [\tau, \tau_1]$. Then according to (1) and (24)

$$y(\varphi(t)) = -r^{-\frac{1}{\lambda}}(t)\left|\left(a(t)|y'(t)|^{p-1}y'(t)\right)'\right|^{1/\lambda} \operatorname{sgn}\left(a(t)|y'(t)|^{p-1}y'(t)\right)' = 0 \tag{26}$$

for $t \in I$. As $\varphi(\tau_1) > \tau$ we have

$$[\varphi(\tau), \tau] \subset [\varphi(\tau), \varphi(\tau_1)] \subset \{\varphi(t) \colon t \in I\}\,.$$

From this and from (26), $y(t) = 0$ on $[\varphi(\tau), \tau]$ that contradicts (25). Hence, $\varphi(\tau) = \tau$.

We prove that y is oscillatory. Suppose, contrarily, that $y(t) > 0$ in a left neighbourhood of τ; case $y(t) < 0$ can be studied similarly. From this and from $\varphi(\tau) = \tau$ an interval $I_1 = [\tau_2, \tau)$, $\tau_2 < \tau$ exists such

$$y\big(\varphi(t)\big) > 0 \quad \text{for} \quad t \in I_1 \,. \tag{27}$$

As, according to (2), y_2 is decreasing on I_1 and (24) implies $y_2(\tau) = 0$ we have $y_2 > 0$ on I_1; hence, $y' > 0$ on I_1. The contradiction with (27) and (24) proves that y is oscillatory.

Case (i). The proof follows from Theorem A(iii) by the same way as in the proof of Theorem 1.

Case (ii). Let $\lambda \geq p$ and $R \in C^1(\mathbb{R}_+)$. Then (i) implies φ is nontrivial in any left neighbourhood of τ. Suppose that an increasing sequence $\{\tau_k\}_{k=1}^{\infty}$ exists such that $\lim\limits_{k\to\infty} \tau_k = \tau$ and $\varphi(\tau_k) = \tau_k$. As φ is nondecreasing in J, $\{\tau_k\}$ may be choosen such that

$$\varphi(t) \in [\tau_k, \tau] \quad \text{for} \quad t \in [\tau_k, \tau]\,. \tag{28}$$

It follows from (24) and (25) that $y_2(\tau) = 0$ and $F(\tau) = 0$. Denote $\bar{F}_k = \max\limits_{\tau_k \leq s \leq \tau} F(s)$. Then (28), (7) and (9) imply

$$\begin{aligned} F(s) = -\int_s^{\tau} F'(\sigma)\, d\sigma \leq \bar{F}_k \int_{\tau_k}^{\tau} \frac{|R'(\sigma)|}{R(\sigma)}\, d\sigma \\ + 2\delta\gamma^{-\lambda} \bar{F}_k^{\omega} \int_{\tau_k}^{\tau} a^{-\frac{1}{p}}(\sigma) R^{\frac{1}{p+1}}(\sigma)\, d\sigma \end{aligned}$$

for $s \in [\tau_k, \tau]$ where $\omega = \frac{1}{p+1} + \frac{\lambda}{\lambda+1} \geq 1$ due to $\lambda \geq p$. Hence,

$$\bar{F}_k \leq \bar{F}_k \int_{\tau_k}^{\tau} \frac{|R'(\sigma)|}{R(\sigma)}\, d\sigma + 2\delta\gamma^{-\lambda} \bar{F}_k^{\omega} \int_{\tau_k}^{\tau} a^{-\frac{1}{p}}(\sigma) R^{\frac{1}{p+1}}(\sigma)\, d\sigma \tag{29}$$

$k = 1, 2, \ldots$. As $\lim\limits_{k\to\infty} \bar{F}_k = F(\tau) = 0$ and

$$\lim_{k\to\infty} \int_{\tau_k}^{\tau} \frac{|R'(\sigma)|}{R(\sigma)}\, d\sigma = 0, \quad \lim_{k\to\infty} \int_{\tau_k}^{\tau} a^{-\frac{1}{p}}(\sigma) R^{\frac{1}{p+1}}(\sigma)\, d\sigma = 0$$

we obtain the contradiction in (29) for large k. Hence, $\{\tau_k\}$ does not exists and the statement holds in this case. □

The following result is a consequence of Theorem 2 and Theorem 4.

Theorem 5. *If $\varphi(t) < t$ on $\mathbb{R}_+$, then all solutions of (1) are proper.*

Lemma 3. *Let y be a singular solution of the 1st kind, let $T \in [0, \tau)$ be such that*

$$\int_T^{\tau} R^{-1}(t) |R'(t)|\, dt \leq \frac{1}{2}\,, \tag{30}$$

$I = [T, \tau]$, $K > 0$, $\omega \geq 0$ and $|e(t)| \leq K(\tau - t)^{\omega}$ on I. Then

$$F(t) \leq K_1 (\tau - t)^{\delta(\omega+1)}\,, \qquad t \in I$$

where $K_1 = \left[2\delta(\omega+1)^{-1} K \max\limits_{0 \leq \sigma \leq \tau} a^{-\frac{1}{p}}(\sigma) R^{\frac{1}{p+1}}(\sigma)\right]^{\delta}$.

Proof. Let y be a singular solution of the 1st kind. Then (9) implies

$$R^{-1}(t)|y_2(t)|^\delta \leq F(t)\,, \qquad |y'(t)| \leq C\, F^{\frac{1}{p+1}}(t)$$

on I with $C = \max_{t\in I} a^{-\frac{1}{p}}(t) R^{\frac{1}{p+1}}(t) > 0$. Define $\bar{F}(t) = \max_{s\in[t,\tau]} F(s)$ for $t \in I$. From this and from (7), (8) and (30)

$$\begin{aligned} F(s) &= -\int_t^\tau F'(\sigma)\, d\sigma \leq \int_t^\tau R^{-1}(\sigma)|R'(\sigma)|F(\sigma)\, ds + \delta \int_t^\tau |y'(\sigma)e(\sigma)|\, ds \\ &\leq \bar{F}(t) \int_T^\tau R^{-1}(\sigma)|R'(\sigma)|\, ds + C_1 \int_t^\tau F^{\frac{1}{p+1}}(\sigma)(\tau-\sigma)^\omega ds \\ &\leq \frac{\bar{F}(t)}{2} + \frac{C_1}{\omega+1} \bar{F}^{\frac{1}{p+1}}(t)(\tau-t)^{\omega+1} \end{aligned}$$

for $t \in I$ and $t \leq s \leq \tau$ where $C_1 = \delta K C$. Hence,

$$\bar{F}(t) \leq \frac{\bar{F}(t)}{2} + \frac{C_1}{\omega+1} \bar{F}^{\frac{1}{p+1}}(t)(\tau-t)^{\omega+1}$$

or

$$F(t) \leq \bar{F}(t) \leq K_1(\tau-t)^{\delta(\omega+1)} \quad \text{on } I.$$

$\square$

The following theorem gives us an estimate from above of singular solutions of the 1st kind.

Theorem 6. *Let y be a singular solution of* (1) *of the 1st kind and $M > 0$ be such that $\varphi'(t) \leq M$ in a left neighbourhood S of τ.*

(i) *Let $\lambda \geq p$ and $m > 0$. Then a positive constant K and a left neighbourhood J of τ exist such that*

$$|y(t)| \leq K(\tau-t)^m\,, \qquad |y_2(t)| \leq K(\tau-t)^{\frac{(\lambda+1)m}{p+1}} \quad \textit{on } J.$$

(ii) *Let $\lambda < p$ and $\varepsilon > 0$. Then a positive constant K and a left neighbourhood J of τ exist such that*

$$|y(t)| \leq K(\tau-t)^{\frac{p+1}{p-\lambda}-\varepsilon}\,, \qquad |y_2(t)| \leq K(\tau-t)^{\frac{p(\lambda+1)}{p-\lambda}-\varepsilon} \quad \textit{on } J.$$

Proof. Let y be a singular solution of the 1st kind. According to Theorem 4 $\varphi(\tau) = \tau$. Moreover, $\lim_{t\to\tau-} y(t) = \lim_{t\to\tau-} y_2(t) = 0$ and an interval $I = [T, \tau] \subset S$, $0 \leq T_1 < T$ exists such that (30) and

$$|y(t)|^\lambda \leq \frac{1}{2}\,, \quad |y(\varphi(t))|^\lambda \leq \frac{1}{2} \quad \text{for} \quad t \in I\,.$$

Hence, (8) implies $|e(t)| \leq 1$ on I and it follows from Lemma 3 (with $I = I$, $K = 1$, $\omega = 0$)

$$F(t) \leq K(T-t)^\delta\,, \qquad t \in I \tag{31}$$

with

$$K = \left[2\delta \max_{0\leq\sigma\leq T} a^{-\frac{1}{p}}(\sigma) R^{\frac{1}{p+1}}(\sigma)\right]^\delta. \tag{32}$$

Let $\{I_n\}_{n=1}^{\infty}$ be such that $I_1 = I$, $I_n = [T_n, \tau]$, $T_n < T_{n+1} < \tau$ and $\varphi(t) \in I_n$ for $t \in I_{n+1}$, $n = 1, 2, \dots$; this sequence exists due to $\varphi(t) \le t$ and $\varphi(\tau) = \tau$.

We prove the estimate

$$F(t) \le K_n(\tau - t)^{\omega_n} \quad \text{on} \quad I_n \tag{33}$$

by the mathematical induction, where

$$\omega_1 = \delta\,, \quad \omega_{n+1} = \delta\Big[\frac{\lambda}{\lambda+1}\omega_n + 1\Big]\,, \qquad n = 1, 2, \dots \tag{34}$$

and

$$K_1 = K\,, \quad K_{n+1} = K\Big[\gamma^{-\frac{\lambda}{\lambda+1}}\Big(1 + \frac{\lambda}{\lambda+1}\omega_n\Big)^{-1}\big(1 + M^{\omega_n\frac{\lambda}{\lambda+1}}\big)K_n^{\frac{\lambda}{\lambda+1}}\Big]^{\delta}, \quad n = 1, 2, \dots$$

For $n = 1$ (33) follows from (31) and (32). Suppose the validity of (33) for n. Then (6) and (33) imply

$$|y(t)|^{\lambda} \le \big(\gamma^{-1}F(t)\big)^{\frac{\lambda}{\lambda+1}} \le \gamma^{-\frac{\lambda}{\lambda+1}}K_n^{\frac{\lambda}{\lambda+1}}(\tau - t)^{\frac{\lambda}{\lambda+1}\omega_n}\,, \quad t \in I_n$$

and

$$|y(\varphi(t))|^{\lambda} \le \gamma^{-\frac{\lambda}{\lambda+1}}K_n^{\frac{\lambda}{\lambda+1}}M^{\frac{\lambda}{\lambda+1}\omega_n}(\tau - t)^{\frac{\lambda}{\lambda+1}\omega_n}\,, \quad t \in I_{n+1}$$

as

$$0 \le \tau - \varphi(t) = \varphi(\tau) - \varphi(t) = \varphi'(\xi)(\tau - t) \le M(\tau - t)\,, \quad \xi \in [t, \tau]\,.$$

From this and from (8)

$$|e(t)| \le \gamma^{-\frac{\lambda}{\lambda+1}}K_n^{\frac{\lambda}{\lambda+1}}\big[1 + M^{\frac{\lambda}{\lambda+1}\omega_n}\big](\tau - t)^{\frac{\lambda}{\lambda+1}\omega_n} = L_n(\tau - t)^{w_n}\,,$$

where

$$w_n = \frac{\lambda}{\lambda+1}\omega_n \quad \text{and} \quad L_n = \gamma^{-\frac{\lambda}{\lambda+1}}K_n^{\frac{\lambda}{\lambda+1}}\big[1 + M^{w_n}\big]\,.$$

Now, we use Lemma 3 with $I = I_{n+1}$, $K = L_n$ and $\omega = w_n$ and we obtain $F(t) \le K_{n+1}(\tau - t)^{\omega_{n+1}}$. Hence, (33) holds for all $n = 1, 2, \dots$ Denote by

$$z = \frac{\lambda(p+1)}{(\lambda+1)p}\,. \tag{35}$$

We prove that

$$\begin{aligned} &\omega_n \le \delta\frac{1 - z^n}{1 - z}\,, \quad n = 1, 2\dots \quad \text{for} \quad z \ne 1 \\ &\omega_n = \delta n \qquad \text{for} \quad z = 1\,. \end{aligned} \tag{36}$$

If $v_n = \frac{\omega_n}{\delta}$, then (34) implies $v_1 = 1$, $v_{n+1} = zv_n + 1$, $n = 1, \dots$ Hence, $v_n = 1 + z + z^2 + \dots z^{n-1} = \frac{1 - z^n}{1 - z}$ in case $z \ne 1$ and $v_n = n$ in case $z = 1$. Now, (36) follows from this.

We have from (35) that

$$z > 1 \Leftrightarrow \lambda > p\,, \quad z = 1 \Leftrightarrow \lambda = p\,, \quad z < 1 \Leftrightarrow \lambda < p\,.$$

Furthermore, from this and from (36) $\lim\limits_{n\to\infty} \omega_n = \infty$ in case $\lambda \ge p$ and $\lim\limits_{n\to\infty} \omega_n = \frac{\delta}{1-z} = \frac{(p+1)(\lambda+1)}{p-\lambda}$ in case $\lambda < p$. Hence, the statement follows from (33) and (6). $\square$

Acknowledgement. The research was supported by the Grant 201/11/0768 of the Grant Agency of the Czech Republic.

References

[1] I. V. Astashova, *Asymptotic behaviour of solutions of some nonlinear differential equations*, Dokl. Raz. Zas. Sem. Inst. Prikl. Mat. Tbilisi Univ., 1 (1985), 1–11.

[2] V. I. Astashova, *Uniform estimates for positive solutions of quasi-linear ordinary differential equations* (English. Russian original) Izv. Math. **72** No. 6, 1141–1160 (2008); translation from Izv. Ross. Akad. Nauk, Ser. Mat. **72**, No. 6, 85–104 (2008).

[3] V. I. Astashova, *Application of dynamical systems to the study of solutions to nonlinear higher-order differential equations*, J. Math. Sci. (New York) **126**, No. 5, 1361–1391 (2005); translation from Sovrem. Mat. Prilozh. **8** (2003), 3–33.

[4] M. Bartušek, *On noncontinuable solutions of differential equations with delay*, EJQTDE, Spec. Ed. I, 2009, No. 6 (2009), 1-16.

[5] M. Bartušek, *Singular solutions for the differential equations with p-Laplacian*, Arch. Math. (Brno) 41 (2005), 123–128.

[6] M. Bartušek, J. R. Graef, *Strong nonlinear limit-point/limit-circle properties for second order differential equations with delay*, PanAmer. Math. J. 20 (2010), 31–49.

[7] M. Bartušek, M. Medveď, *Existence of global solutions for systems of second-order functional-differential equations with p-Laplacian*, EJDE 2008, No. 40 (2008), 1–8.

[8] M. Bartušek, J. R. Graef, *Limit-point/limit-circle problem II*, PanAmer. Math. J., to appear.

[9] C. V. Cofmann, D. F. Ulrich, *On the continuation of solutions of a certain non-linear differential equations*, Monatsh. Math. B71 (1967), 385–392.

[10] J. Hale, *Theory of functional differential equations*, Springer-Verlag, New York, Heidelberg, Berlin, 1977.

[11] J. W. Heidel, *Uniqueness, continuation and nonoscillation for a second order differential equation*, Pacific J. Math. 32 (1970), 715–721.

[12] I. T. Kiguradze, *Asymptotic properties of solutions of a nonlinear differential equation of Emden-Fowler type*, Izv. Akad. Nauk SSSR Ser. Mat. 29 (1965), 965–986.

[13] I. Kiguradze, T. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer, Dordrecht–Boston–London 1993.

[14] M. Medveď, E. Pekárková, *Existence of noncontinuable solutions of second order differential equations with p-Laplacian*, EJDE 2007, No. 136 (2007), 1–9.

[15] D. Mirzov, *Asymptotic properties of solutions of systems of nonlinear nonautonomous ordinary differential equations*, Folia Fac. Sci. Natur. Univ. Masaryk. Brun., Math 14, 2004.

[16] E. Pekárková, *Estimations of noncontinuable solutions of second order differential equations with p-Laplacian*, Arch. Math. (Brno) 46 (2010), No. 2, 135–144.

(Received September 20, 2011)

Faculty of Science, Masaryk University Brno, Kotlářská 2, 611 37 Brno, The Czech Republic

E-mail address: bartusek@math.muni.cz